Gallot-Tanno theorem for pseudo-Riemannian metrics and a proof that decomposable cones over closed complete pseudo-Riemannian manifolds do not exist.

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Introduction. Let g be a Riemannian or pseudo-Riemannian metric

$$g = \sum_{i,j=1}^{n} g_{ij}(x_1, ..., x_n) dx_i dx_j$$

on an $n-$dimensional manifold M. We consider the following equation on the unknown function λ on M.

$$\nabla_k \nabla_j \nabla_i \lambda + 2\nabla_k \lambda \cdot g_{ij} + \nabla_i \lambda \cdot g_{jk} + \nabla_j \lambda \cdot g_{ik} = 0. \tag{1}$$

This equation is a famous one; it naturally appeared in different parts of differential geometry. Couty [2] and De Vries [11] studied it in the contex of conformal transformations of Riemannian metrics. They showed that, under certan additional assumptions, conformal vector fields generate nonconstant solutions of the equation (1).

The equation also appears in investigation of geodesically equivalent metrics. Recall that two metrics on one manifold are *geodesically equivalent*, if every geodesic of one metric is a reparametrized geodesic of the second metric. Solodovnikov [9] has shown that Riemannian metrics on $(n > 3)-$dimensional manifolds admitting nontrivial 3-parameter family of geodesically equivalent metrics allow nontrivial solutions of (a certain generlaization of) (1). Recently, this result was generalised for pseudo-Riemannian metrics [6, Corollary 4]. Moreover, as it was shown in [5, Corollary 3] (see also [4]), an Einstein manifold of nonconstant scalar curvature admitting nontrival geodesic equivalence, after a proper scaling, admits a nonconstant solution of (1). Tanno [10] (see also [4]) related the equation (1) to projective vector fields, i.e., to vector fields whose local flows take unparametrized geodesics to geodesics. He has shown that every nonconstant solution λ of this equation allows to construct a nontrivial projective vector field.

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Obata used this equation trying to understand the relation between the eigenvalues of the laplacian Δ_g and the geometry and topology of the manifold. He observed [8] that the eigenfunctions corresponding to the second eigenvalue of the Laplacian of the metrics of constant curvature $+1$ on the sphere satisfy the equation (1), and asked the question whether the existence of a nonconstant solution of this equation on a complete manifold implies that the manifold is covered by the sphere with the standard metric. The positive answer to this question was indepedently and simultaneously obtained by Gallot [3] and Tanno [10].

This note generalizes the result of Gallot [3] and Tanno [10] to pseudo-Riemannian metrics:

Theorem 1. *Let g be a light-line-complete connected pseudo-Riemannian metric of indefinite signature (i.e., for no constant c the metric $c \cdot g$ is Riemannian) on a closed $n-$dimensional manifold M^n. Then, every solution of* (1) *is constant.*

Theorem 2. *Let g be a negative-definite metric (i.e., $-g$ is a Riemannian metric) on a closed connected manifold M. Then, every solution of* (1) *is constant.*

Example of Alexeevsky, Cortes, Galaev and Leistner [1, Example 3.1] combined with Lemma 2 below shows that in the pseudo-Riemannian case the assumption that the metric is complete (but the manifold is not closed) is not sufficient to ensure that every solution of (1) is constant.

The equation (1) naturally appears also in the investigation of the holonomy group of cones over pseudo-Riemannian manifolds. Recall that the *cone over* (M^n, g) is the pseudo-Riemannian manifold $(\hat{M}^{n+1}, \hat{g})$, where $\hat{M} = \mathbb{R}_{>0} \times M$ and

$$\hat{g} = (dx_0)^2 + x_0^2 \cdot \left(\sum_{i,j=1}^{n} g_{ij}(x_1, ..., x_n) dx_i dx_j \right), \tag{2}$$

where x_0 is the standard coordinate on $\mathbb{R}_{>0}$ and $x_1, ..., x_n$ are local coordinates on M^n. Following [1, 3], we will show that the decomposability of the cone (i.e., the existence of a proper nondegenerate subspace $U \subset T_p\hat{M}$ invariant with respect to the holonomy group) implies the existence of an nonconstant solution of (1) on (M, g), see Lemma 2 below. Combining this with Theorems 1, 2, we obtain

Corollary 1. *Let g be a light-line-complete pseudo-Riemannian metric of indefinite signature on a closed $n-$dimensional manifold M^n. Then, the corresponding cone $(\hat{M}, \hat{g})$ is not decomposable.*

Corollary 2. *Let g be a complete negative-definite pseudo-Riemannian metric on a closed $n-$dimensional manifold M^n. Then, the corresponding cone $(\hat{M}, \hat{g})$ is not decomposable.*

A partial case of Corollaries 1, 2 is [1, Theorem 6.1]. Our proof is different from that of [1] and is shorter. The case when the metric g is Riemannian was solved in [3, Proposition

3.1]: Gallot used the Riemannian version of Theorems 1, 2 to show that if the cone $(\hat{M}, \hat{g})$ over complete Riemannian (M, g) is decomposable, then g has constant curvature $+1$.

Proof of Theorem 1. Let g be an indefinite pseudo-Riemannian metric on M^n. Suppose the function λ satisfies (1). We take a light-line geodesic $\gamma(t)$ whose velocity vector will be denoted by $\dot{\gamma} = (\dot{\gamma}^i)$, multiply (1) by $\dot{\gamma}^i\dot{\gamma}^j\dot{\gamma}^k$, and sum over i, j, k. Since the geodesic is light-line, at every point $\gamma(t)$ we have

$$\sum_{i,j=1}^{n} g_{ij}\dot{\gamma}^i\dot{\gamma}^j = \sum_{i,k=1}^{n} g_{ik}\dot{\gamma}^i\dot{\gamma}^k = \sum_{j,k=1}^{n} g_{jk}\dot{\gamma}^j\dot{\gamma}^k \equiv 0 \quad \text{implying} \quad \sum_{i,j,k=1}^{n} \dot{\gamma}^i\dot{\gamma}^j\dot{\gamma}^k\nabla_k\nabla_j\nabla_i\lambda = 0.$$

By definition of the geodesic, $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$ implying

$$\sum_{i,j,k=1}^{n} \dot{\gamma}^i\dot{\gamma}^j\dot{\gamma}^k\nabla_k\nabla_j\nabla_i\lambda = \sum_{k=1}^{n} \dot{\gamma}^k\nabla_k\left(\sum_{j=1}^{n} \dot{\gamma}^j\nabla_j\left(\sum_{i=1}^{n} \dot{\gamma}^i\nabla_i\lambda\right)\right) = \tfrac{d^3}{dt^3}\lambda(\gamma(t))$$

implying $\frac{d^3}{dt^3}\lambda(\gamma(t)) = 0$ implying that $\lambda = \mathrm{const}_2 t^2 + \mathrm{const}_1 t + \mathrm{const}_0$.

But by assumption the manifold M is compact implying that the function λ is bounded, and the function $\mathrm{const}_2 t^2 + \mathrm{const}_1 t + \mathrm{const}_0$ is bounded if and only if $\mathrm{const}_2 = \mathrm{const}_1 = 0$. Then, λ is constant along every light-line geodesic. Since every two points of a connected pseudo-Riemannian manifold of indefinite signature can be connected by a sequence of light-line geodesics, the function λ is a constant. Theorem 1 is proved.

Proof of Theorem 2. We multiply (1) by g^{ij} and sum over $i, j \in 1, ..., n$. We obtain: $\nabla_k(\Delta_g\lambda) = -2(n+1)\nabla_k\lambda$, where $\Delta_g := \sum_{i,j=1}^{n} g^{ij}\nabla_i\nabla_j : C^\infty(M) \to C^\infty(M)$ is the laplacian of g. Then, for a certain constant C we have $\Delta_g(\lambda + C) = -2(n+1)(\lambda + C)$. Thus, $\lambda + C$ is an eigenfunction of Δ_g with negative eigenvalue $-2(n+1)$. Since the metric g is negative-definite and the manifold is closed, laplacian of g is positive definite on nonconstant functions implying $\lambda + C \equiv \mathrm{const}$. Thus, λ is constant. Theorem 2 is proved.

Proof of Corollaries 1, 2. It is well-known that if a manifold $(\hat{M}, \hat{g})$ is decomposable, then there exists a symmetric tensor $\hat{a} = (\hat{a}_{ij})$, $i, j = 0, ..., n$ such that $\hat{a} \neq \mathrm{const}\cdot\hat{g}$ for every $\mathrm{const} \in \mathbb{R}$ and such that its covariant derivative vanishes: $\hat{\nabla}_k\hat{a}_{ij} \equiv 0$. We denote by μ the $(0,0)-$componenent of $\hat{a}$, by λ_i the $(0,i)-$component of $\hat{a}$ (the symmetric $(i,0)-$component is also λ_i), and by a_{ij} the $(i,j)-$component of $\hat{a}$ for $i, j = 1, ..., n$, so that the matrix of $\hat{a}$ is

$$(\hat{a}_{ij}) = \begin{pmatrix} \mu & \lambda_1 & \dots & \lambda_n \\ \lambda_1 & a_{11} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ \lambda_n & a_{n1} & \dots & a_{nn} \end{pmatrix} \tag{3}$$

The components of μ, λ_i, a_{ij} can a priori depend on t. For a fixed t (say, for $t = 1$), one can view μ, λ_i, a_{ij} as geometrical objects on M: μ is a function on M, λ_i is an $(0,1)-$tensor on M, and a_{ij} is a symmetric $(0,2)-$tensor on M (i.e., if we change the local coordinate system on M the componenents of λ_i and a_{ij} change according to the tensor rules). We will denote by ∇ ($\hat\nabla$, resp.) the covariant derivative in the sense of g ($\hat g$, resp.) and by Γ^k_{ij} ($\hat\Gamma^k_{ij}$, resp.) the corresponding Christoffel symbols. We will need the following

Lemma 1. *Let $\hat a$ given by (3) satisfy $\hat\nabla \hat a = 0$. Then, the tensors λ_i, a_{ij}, and the function μ on M satisfy (we assume $t = 1$)*

$$\begin{aligned} \nabla_k a_{ij} &= -\lambda_i g_{jk} - \lambda_j g_{ik}, && (4)\\ \nabla_j \lambda_i &= a_{ij} - \mu g_{ij}, && (5)\\ \nabla_i \mu &= 2\lambda_i. && (6) \end{aligned}$$

Proof. Let us calculate $\hat\Gamma^i_{jk}$ in terms of g_{ij} and Γ^i_{jk} at the point $(1, x_1, ..., x_n)$ of the cone $\hat M$: substututing (2) in $\hat\Gamma^i_{jk} = \frac{1}{2}\sum_{h=0}^n \hat g^{ih}\left(\partial_k \hat g_{jh} + \partial_j \hat g_{hk} - \partial_h \hat g_{jk}\right)$ we obtain

$$\begin{array}{lll|lll} \hat\Gamma^0_{j0} = \hat\Gamma^0_{0j} = 0 & & \forall j \in 0, ..., n & \hat\Gamma^0_{jk} = -g_{jk} & & \forall j, k \in 1, ..., n\\ \hat\Gamma^j_{j0} = \hat\Gamma^j_{0j} = 1 & & \forall j \in 1, ..., n & \hat\Gamma^i_{j0} = \hat\Gamma^i_{0j} = 0 & & \forall i \neq j \in 1, ..., n.\\ \hat\Gamma^i_{jk} = \Gamma^i_{jk} & & \forall i, j, k \in 1, ..., n & & & \end{array} \qquad (7)$$

Substituting (3) and (7) in the equation $\hat\nabla_k \hat a_{ij} = 0$, we obtain that for every $i, j, k \in 1, ..., n$

$$0 = \hat\nabla_k \hat a_{ij} = \partial_k a_{ij} - \hat\Gamma^0_{kj}\hat a_{i0} - \hat\Gamma^0_{ik}\hat a_{0j} - \sum_{h=1}^n \left[\hat\Gamma^h_{kj}\hat a_{ih} + \hat\Gamma^h_{ik}\hat a_{hj}\right] = \nabla_k a_{ij} + g_{kj}\lambda_i + g_{ik}\lambda_j,$$

which proves (4). Similarly, substituting (3) and (7) in $\hat\nabla_j \hat a_{i0} = 0$ we obtain (5), and substituting (3) and (7) in $\hat\nabla_i \hat a_{00} = 0$ we obtain (6). Lemma 1 is proved.

Lemma 2. *The $(0,1)-$tensor λ_i is the differential of a certain function λ on M, i.e., $\lambda_i = \nabla_i\lambda = \partial_i\lambda$. Moreover, the function λ satisfies the equation (1). Moreover, if λ is constant, then $\hat a$ is proportional to $\hat g$ (with a constant coefficient of proportionality).*

Proof. We multiply (4) by g^{ij} (which is the dual tensor to g_{ij}: $\sum_{h=1}^n g^{ih}g_{hj} = \delta^i_j$) and sum over i and j: since $\nabla_k g^{ij} = 0$ we obtain $\nabla_k \sum_{i,j=1}^n a_{ij}g^{ij} = -2\lambda_k$. Thus, $\lambda_k = \nabla_k\left(-\frac{1}{2}\sum_{i,j=1}^n a_{ij}g^{ij}\right) = \nabla_k\lambda$ for the function $\lambda := -\frac{1}{2}\sum_{i,j=1}^n a_{ij}g^{ij}$. Now, covariantly differentiating (5), replacing λ_i by $\nabla_i\lambda$ and replacing the covariant derivatives of a_{ij} and μ using (4) and (6) we obtain

$$\begin{aligned} 0 = &\ \nabla_k\left(\nabla_j\lambda_i - a_{ij} + \mu g_{ij}\right) = \nabla_k\nabla_j\nabla_i\lambda - \nabla_k a_{ij} + \nabla_k\mu \cdot g_{ij}\\ = &\ \nabla_k\nabla_j\nabla_i\lambda + \nabla_i\lambda \cdot g_{jk} + \nabla_j\lambda \cdot g_{ik} + 2\nabla_k\lambda \cdot g_{ij}, \end{aligned}$$

which is the equation (1).

If λ is constant, μ is constant by (6). Then, (5) implies $a = \mu \cdot g$. Since $\lambda_i = \partial_i \lambda = 0$, we have $\hat{a} = \mu \cdot \hat{g}$, i.e., $\hat{a}$ is proportional to $\hat{g}$ at every point of $\hat{M}$ with $t = 1$. Since $\hat{a}$ and $\hat{g}$ are covariantly constant, $\hat{a}$ is proportional to $\hat{g}$ at every point of $\hat{M}$. Lemma 2 is proved.

Remark 1. Corollaries 1, 2 easily follow from Theorems 1, 2 and Lemma 2.

Certain generalizations. One can easily generalize our proof of Theorem 1 for higher Gallot equations E_p introduced in [3, Section 4]: for every $p \in \mathbb{N}$ the equation E_p is

$$D^{p+1} f(Y_1, ..., Y_{p+1}) + \sum_{1 \le s \le \frac{p+1}{2}} \sum_{\sigma \in S_{p+1}} \lambda(s,\sigma) \left(g^s \otimes D^{p+1-2s} f\right) (Y_{\sigma(1)}, ..., Y_{\sigma(p+1)}) = 0, \quad (8)$$

where f is the unknown function, S_{p+1} denotes the set of all permutations of $\{1, ..., p+1\}$, $\lambda(s,\sigma)$ denotes certain numbers depending on $s \in 1, ..., [\frac{p+1}{2}]$ and on $\sigma \in S_{p+1}$ whose precise values are not important for our proof, $Y_1, ..., Y_{p+1}$ are arbitrary vector fields, and D^k denotes the $k-$th covariant derivative (so for example $D^2 f(X,Y) = \sum_{i,j=1}^n X^i Y^j \nabla_j \nabla_i f$).

Theorem 3. *Let g be a light-line-complete connected pseudo-Riemannian metric of indefinite signature on a closed $n-$dimensional manifold M^n. Then, every solution of* (8) *is constant.*

Proof. We take a light-line geodesic γ, and take arbitrary vector fields Y_i such that at every point of the geodesic γ we have $Y_i = \dot{\gamma}$. Since $g(\dot{\gamma}, \dot{\gamma}) = 0$ and $s \geq 1$, we obtain $\left(g^s \otimes D^{p+1-2s} f\right) (Y_{\sigma(1)}, ..., Y_{\sigma(p+1)}) = 0$. Then, $D^{p+1}(\dot{\gamma}, ..., \dot{\gamma}) = 0$ implying $\frac{d^{p+1}}{dt^{p+1}} f(\gamma(t)) \equiv 0$ implying $f = \mathrm{const}_p t^p + ... + \mathrm{const}_0$. Since the manifold is compact, the function f must be bounded implying $\mathrm{const}_p = ... = \mathrm{const}_1 = 0$. Thus, the function f must be constant along every light-line geodesic. Since every two points of a connected pseudo-Riemannian manifold of indefinite signature can be connected by a sequence of light-line geodesics, the function λ is a constant. Theorem 3 is proved.

Another possible generalization is due to the observation that in our proof of Corollaries 1, 2 we actually used the existence of a covariantly-constant symmetric $(0,2)-$tensor $\hat{a}_{ij} \neq \mathrm{const} \cdot \hat{g}_{ij}$ only. Decomposability of the metric $\hat{g}$ implies the existence of such a tensor $\hat{a}$, but not vice versa: in the pseudo-Riemannian case there exist metrics g admitting covariantly-constant symmetric $a \neq \mathrm{const} \cdot g$, see [7]. So, in fact we proved

Corollary 3. *Let g be a light-line-complete pseudo-Riemannian metric of indefinite signature on a closed $n-$dimensional manifold M^n. Then, every symmetric $(0,2)-$tensor $\hat{a}_{ij}$ on the corresponding cone $(\hat{M}, \hat{g})$ such that $\hat{\nabla}_k \hat{a}_{ij} \equiv 0$ is proportional to $\hat{g}_{ij}$.*

Corollary 4. *Let g be a complete negative-definite pseudo-Riemannian metric on a closed $n-$dimensional manifold M^n. Then, every symmetric $(0,2)-$tensor $\hat{a}_{ij}$ on the corresponding cone $(\hat{M}, \hat{g})$ such that $\hat{\nabla}_k \hat{a}_{ij} \equiv 0$ is proportional to $\hat{g}_{ij}$.*

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